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AN ANALYSIS OF PHYSICAL OPTICS MODELS FOR ROUGH SURFACE  
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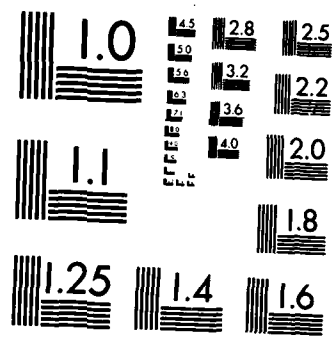
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RADC-TR-84-195  
In-House Report  
September 1984



# ***AN ANALYSIS OF PHYSICAL OPTICS MODELS FOR ROUGH SURFACE SCATTERING***

Robert J. Papa  
John F. Lennon  
Richard L. Taylor

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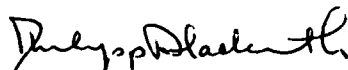
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REPORT DOCUMENTATION PAGE				
1a. REPORT SECURITY CLASSIFICATION Unclassified		1b. RESTRICTIVE MARKINGS N/A		
2a. SECURITY CLASSIFICATION AUTHORITY N/A		3. DISTRIBUTION/AVAILABILITY OF REPORT Approved for public release; Distribution unlimited.		
2b. DECLASSIFICATION/DOWNGRADING SCHEDULE N/A				
4. PERFORMING ORGANIZATION REPORT NUMBER(S) RADC-TR-84-195		5. MONITORING ORGANIZATION REPORT NUMBER(S)		
6a. NAME OF PERFORMING ORGANIZATION Rome Air Development Center	6b. OFFICE SYMBOL (If applicable) EEC	7a. NAME OF MONITORING ORGANIZATION		
6c. ADDRESS (City, State and ZIP Code) Hanscom AFB Massachusetts 01731		7b. ADDRESS (City, State and ZIP Code)		
8a. NAME OF FUNDING/SPONSORING ORGANIZATION Rome Air Development Center	8b. OFFICE SYMBOL (If applicable) EEC	9. PROCUREMENT INSTRUMENT IDENTIFICATION NUMBER N/A		
8c. ADDRESS (City, State and ZIP Code) Hanscom AFB, MA 01731		10. SOURCE OF FUNDING NOS.		
		PROGRAM ELEMENT NO 61102F	PROJECT NO 2305	TASK NO J4
				WORK UNIT NO 07
11. TITLE (Include Security Classification) An Analysis of Physical Optics Models for Rough Surface Scattering				
12. PERSONAL AUTHOR(S) Papa, Robert J., Lennon, John F., Taylor, Richard L.				
13a. TYPE OF REPORT In-House	13b. TIME COVERED FROM 83 Jan TO 84 Jun	14. DATE OF REPORT (Yr., Mo., Day) 1984 September	15. PAGE COUNT 30	
16. SUPPLEMENTARY NOTATION N/A				
17. COSATI CODES		18. SUBJECT TERMS (Continue on reverse if necessary and identify by block number)		
FIELD	GROUP	SUB GR		
2014	0903	1709	Rough Surface Scattering, Physical Optics, Electromagnetics, <i>R</i>	
19. ABSTRACT (Continue on reverse if necessary and identify by block number) Many techniques have been applied to analyze the effects of the environment (specifically rough surface scattering) on the performance of radar and communication systems. Physical Optics is one of the most frequently applied approaches. Several topics related to the use of physical optics scattering models are discussed in this report. These topics include the limitations and constraints that are inherent in the application of physical optics models, the ranges of its validity, different representations of normalized cross sections, $\sigma^\circ$ , and the parameter ranges in which those forms have to be used, and finally the effect of the different representations for $\sigma^\circ$ on the calculation of diffuse scattered power. It is shown that if the Rayleigh parameter dependence of $\sigma^\circ$ is taken into account and the surface shadowing is included, then the physical optics model for electromagnetic scattering can be applied to surfaces that represent an extensive range of roughness conditions. <i>scat. geometry included</i>				
20. DISTRIBUTION/AVAILABILITY OF ABSTRACT UNCLASSIFIED/UNLIMITED <input type="checkbox"/> SAME AS RPT <input checked="" type="checkbox"/> DTIC USERS <input type="checkbox"/>		21. ABSTRACT SECURITY CLASSIFICATION UNCLASSIFIED		
22a. NAME OF RESPONSIBLE INDIVIDUAL Robert J. Papa		22b. TELEPHONE NUMBER (Include Area Code) (617) 861-3735	22c. OFFICE SYMBOL RADC/EECT	

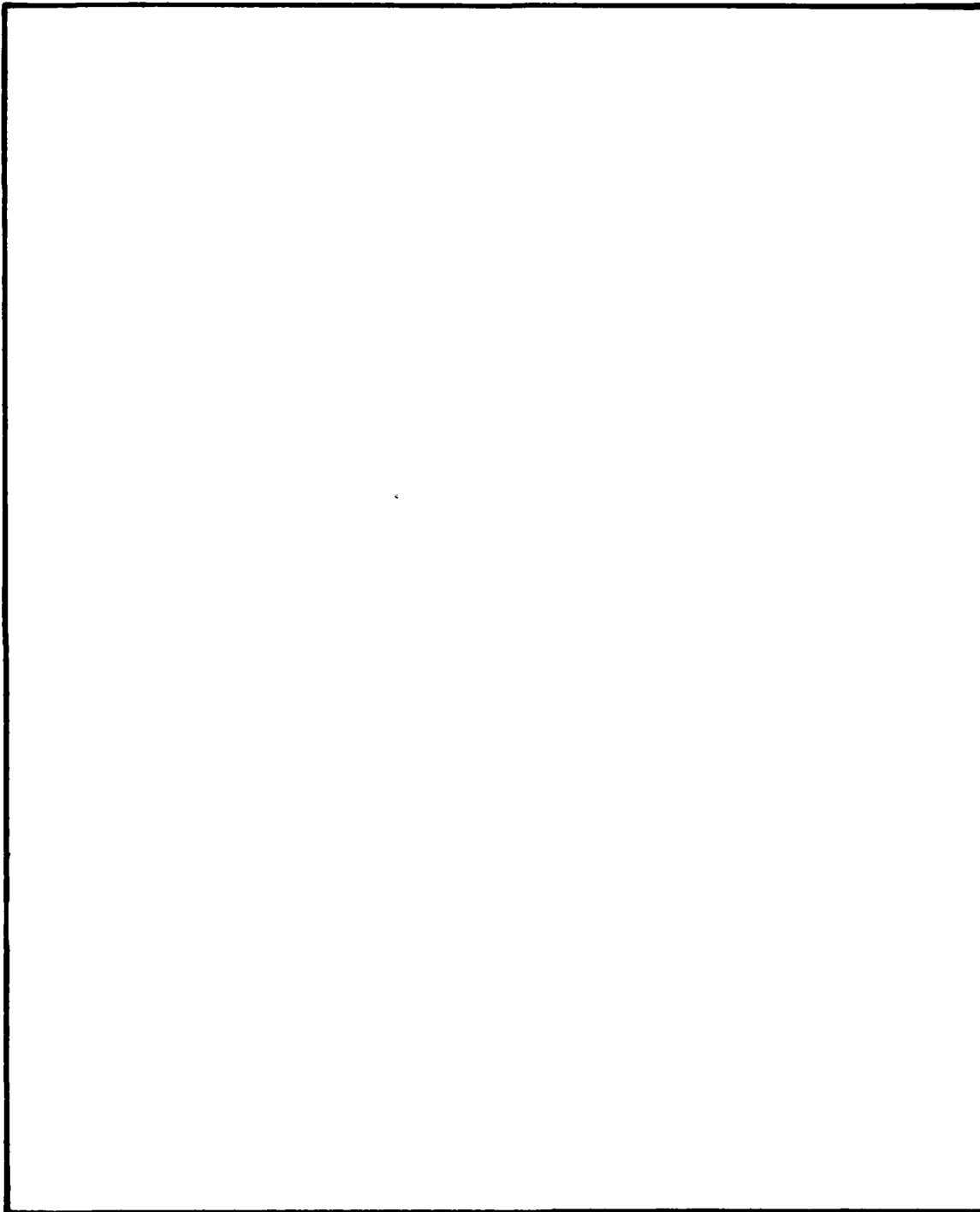
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# An Analysis of Physical Optics Models for Rough Surface Scattering

## 1. INTRODUCTION

There are a large number of approaches to the calculation of the electromagnetic scattering from rough surfaces.<sup>1-6</sup> The importance of these techniques rests in their relevance to the determination of the effects of the real environment on radar and communications systems. The analyses contain certain assumptions about the nature of the rough surface in relation to the electromagnetic phenomena.

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(Received for Publication 18 September 1984)

1. Beckman, P., and Spizzichino, A. (1963) The Scattering of Electromagnetic Waves From Rough Surfaces, Macmillan Co., New York
2. Ruck, G.T., Barrick, D.E., Stuart, W.D., and Krichbaum, C.K. (1970) Radar Cross Section Handbook, Vol. 2, Plenum Press, New York.
3. Long, N.W. (1975) Radar Reflectivity of Land and Sea, Lexington Books, Lexington, Mass.
4. Brown, G.S. (1982) A stochastic Fourier transform approach to scattering from perfectly conducting randomly rough surfaces, IEEE Trans. Antennas Propag. AP-30 (No. 6):1135-1144.
5. Peake, W.H. (1959) The Interaction of Electromagnetic Waves With Some Natural Surfaces, Antenna Laboratory, Ohio State University Report-898-2.
6. Bahar, E. (1981) Scattering cross sections for random rough surfaces: Full wave analysis, Radio Sci. 16:331-341.

Particular emphasis has been placed on the characterization of the surface in terms of the statistical distribution of the heights, their degree of correlation ( $T$  is the surface correlation length), the variance of heights  $\sigma^2$ , and a complex dielectric constant representing a particular terrain type.<sup>7,8</sup> These features are then related to a normalized radar cross section for the terrain through an electromagnetic analysis.<sup>9,10</sup> One of the most frequently applied approaches is that of physical optics.

In this report, we will discuss several topics related to the use of a physical optics scattering model. These are the limitations and constraints that are inherent in the application of such a model, the range of validity of the analyses, how different representations of normalized cross section  $\sigma^\circ$  have to be introduced for different parameter ranges, and the effects of these different representations of the normalized cross section<sup>9</sup> on the calculation of the diffuse scattered power.

## 2. THE CONDITIONS FOR PHYSICAL OPTICS

Over the years, the arguments to support the validity of physical optics have centered around the use of the Kirchhoff integral representation for the scattered EM wave, where the boundary conditions on the surface have been satisfied through the use of the Fresnel plane wave reflection coefficients. The boundary conditions are met by specifying the total field on the rough surface as the sum of an incident field and a scattered field. The scattered field is expressed as the product of the incident field and the Fresnel plane wave reflection coefficient. The use of the Fresnel plane wave reflection coefficient is justified if the local radii of curvature on the rough surface are large compared to a wavelength ( $R_c \gg \lambda$ ). We wish to relate this constraint to more readily handled surface parameters.

In Ulaby et al.<sup>11</sup> there are two conditions given for the validity of the Kirchhoff approximation:  $kT > 6$  and  $T^2/\sigma > 2.76 \lambda$ , where  $k = 2\pi/\lambda$ ,  $\lambda$  = EM wavelength,

7. Papa, R. J., and Lennon, J. F. (1980) Electromagnetic scattering from rough surfaces based on statistical characterization of the terrain, International Radio Science Symposium, (URSI), Quebec, Canada.
8. Papa, R. J., Lennon, J. F., and Taylor, R. L. (1980) Electromagnetic Wave Scattering From Rough Terrain, RADC-TR-80-300, AD A098939.
9. Papa, R. J., Lennon, J. F., and Taylor, R. L. (1982) Further Considerations in Models of Rough Surface Scattering, RADC-TR-82-326, AD A130424.
10. Papa, R. J., Lennon, J. F., and Taylor, R. L. (1983) Multipath effects on an azimuthal monopulse system, IEEE Trans. on Aerospace and Electronic Systems, pp. 585-597.
11. Ulaby, F. T., Moore, R. K., and Fung, A. K. (1982) Microwave Remote Sensing, Vol. II, Addison-Wesley.

$T$  = surface correlation length, and  $\sigma$  = standard deviation in surface height. In the present report, it will be shown that a single condition ( $T \gg \lambda$ ) is sufficient for the validity of physical optics. It will be shown that for a rough surface with a Gaussian correlation function,  $T \gg \lambda$  implies that the radius of curvature  $R_c$  must be large compared to a wavelength ( $R_c \gg \lambda$ ). Thus, the second condition given by Ulaby et al<sup>11</sup> is redundant.

## 2.1 Magnetic Field Integral Equation

To clarify some of the implications in the use of physical optics concepts we follow the treatment of Brown<sup>4</sup> who starts with the magnetic field integral equation for the current  $J_s$  induced on a perfectly conducting rough surface by an incident magnetic field  $\vec{H}^i$ .

$$\vec{J}_s(\vec{r}) = 2\vec{n} \times \vec{H}^i(\vec{r}) + \frac{1}{2\pi} \int_{S_0} \vec{n}(\vec{r}) \times [\vec{J}_s(\vec{r}_0) \times \vec{\nabla}_0 g(|\vec{r} - \vec{r}_0|)] dS_0, \quad (1)$$

where  $\vec{n}$  is the unit normal to the rough surface,  $S_0$  is described by  $z_0 = \xi(x_0, y_0)$ , and  $g(|\vec{r} - \vec{r}_0|)$  is proportional to the free space Green's function:

$$g(|\vec{r} - \vec{r}_0|) = \exp(ik|\vec{r} - \vec{r}_0|)/|\vec{r} - \vec{r}_0|$$

where  $\vec{r}_0$  is the vector from the origin to a point on the rough surface:

$\vec{r}_0 = x_0 \vec{u}_x + y_0 \vec{u}_y + z_0 \vec{u}_z$ . Also,  $\xi(x, y)$  is the surface height (a random variable), and  $\xi_x = \partial \xi / \partial x$  and  $\xi_y = \partial \xi / \partial y$  are the x and y components of the surface slope.

Brown<sup>12</sup> has indicated steps that transform this expression into a new form that can be analyzed to arrive at a sufficient condition for the validity of the physical optics model. The details of this expansion are presented in the appendix. As a result of those operations we arrive at a new integral equation describing the current:

$$\begin{aligned} \lim_{n \rightarrow \infty} \left( \frac{1}{2\pi} \right)^{S_n} \int \cdots \int j_q(\beta_1, \beta_2, \dots, \beta_n) [\tilde{p}_1(k_1 - \beta_1, \vec{k}_2 - \beta_2, \dots, \vec{k}_n - \beta_n) \\ - \Gamma_2(\beta_1, k_1, \beta_2, \vec{k}_2, \dots)] d\beta_1 \dots d\beta_n \\ = 2H_0 \left[ c_z^q - i c_x^q \frac{\partial}{\partial k_{2x}} - i c_y^q \frac{\partial}{\partial k_{2y}} \right] \tilde{p}_1(k_1 - k_{1z}, \vec{k}_2, \dots, \vec{k}_n) \end{aligned} \quad (2)$$

12. Brown, G. S. (1983) Private communication.

Here,  $H_0$  = amplitude of the incident magnetic field,  $c_x^q$ ,  $c_y^q$ , and  $c_z^q$  are polarization factors,

$$s_n = \sum_{i=1}^n i$$

and  $\tilde{p}_1$  is the stochastic Fourier transform of the single point probability density function (pdf). The current is represented as

$$\vec{J}_q = j_q \exp \{ i \vec{k}_{i_t} \cdot \vec{r}_t \} \quad \text{for } q = x, y.$$

Here  $k_1, \vec{k}_2, \dots, \vec{k}_n$  and  $\beta_1, \beta_2, \dots, \beta_n$  are stochastic Fourier transform variables. Also,  $k_{i_t}$  is the transverse part of the incident wave vector and  $\vec{r}_t$  lies in the x-y plane. The details of these equations are presented in the appendix. The particular term of interest in this expression is  $\Gamma_2(k_1, \beta_1, \beta_2, \vec{k}_2, \dots, \vec{k}_n)$ .

$$\begin{aligned} & \Gamma_2(\beta_1, k_1, \beta_2, \vec{k}_2, \dots, \vec{k}_n) \\ &= \frac{1}{(2\pi)^2} \iint \left[ -i \frac{\partial \tilde{g}(\Delta \vec{r}_t, \beta_0)}{\partial \Delta x} \frac{\partial}{\partial k_{2x}} - i \frac{\partial \tilde{g}(\Delta \vec{r}_t, \beta_0)}{\partial \Delta y} \frac{\partial}{\partial k_{2y}} + \tilde{g}_\xi(\Delta \vec{r}_t, \beta_0) \right] \\ & \quad \times \tilde{p}_2(k_1 - \beta_0, \beta_0 - \beta_1, \vec{k}_2, \dots, \vec{k}_n) \exp(i \vec{k}_{i_t} \cdot \Delta \vec{r}_t) d\Delta \vec{r}_t d\beta_0 \end{aligned} \quad (3)$$

Here,  $\tilde{g}$  and  $\tilde{g}_\xi$  are the Fourier transforms of the Green's function and its derivative respectively and  $\tilde{p}_2(\dots)$  is the stochastic Fourier transform of the two point pdf. The significance of  $\Gamma_2$  is that under the condition that this term vanishes ( $\Gamma_2 \rightarrow 0$ ) the integral equation then yields the physical optics result (for a perfect conductor) for the current density:

$$\vec{J}_s = 2 \vec{n} \times \vec{H}^i$$

Thus we can determine a sufficient set of conditions for physical optics in this instance by determining a set of conditions that causes  $\Gamma_2 \rightarrow 0$ . The object is to relate dimensions of the surface to the wavelength of the incident field and in this context we will show that the condition  $T/\lambda \rightarrow \infty$  is a sufficient condition for  $\Gamma_2 \rightarrow 0$  and hence for physical optics conditions to prevail.

Since  $\Gamma_2$  is such a complicated expression we simplify the analysis by examining the type of behavior expected for various elements of the  $\Gamma_2$  integrals. The first term that we wish to consider is  $\tilde{p}_2(\dots)$ , the two point stochastic Fourier transform. Brown<sup>4</sup> has given expressions for  $p_2(\xi, \xi_0, \nabla \xi, \nabla \xi_0, \dots, \nabla^n \xi, \nabla^n \xi_0)$  and

$\tilde{p}_2(k_1, \beta_1, \tilde{k}_2, \tilde{\beta}_2, \dots, \tilde{k}_n, \tilde{\beta}_n)$  and we will use these relations in assessing the behavior of the multivariate distribution under the transform operation.

First, we define  $\tau' = \tau/\lambda = \sqrt{(\Delta x)^2 + (\Delta y)^2}/\lambda$  and  $T' = T/\lambda$ . We restrict our analysis to the case where the surface correlation function,  $R(\tau')$  is Gaussian

$$R(\tau') = \sigma^2 \exp [ - (\tau')^2 / (T')^2 ]$$

In addition, the two point probability density

$$p_2(\xi, \xi_0, \nabla \xi, \nabla_0 \xi_0, \nabla^2 \xi_0, \dots, \nabla^n \xi, \nabla_0^n \xi_0; \nabla \tilde{r}_t)$$

is Gaussian with zero-mean variates.

Next, we consider the quantity

$$\langle \xi \xi_0 \rangle = R(\tau') = \sigma^2 \exp [ - (\tau')^2 / (T')^2 ]$$

Also,

$$\langle \xi_0 \frac{\partial \xi}{\partial x} \rangle = \frac{\partial \tau'}{\partial x} \frac{\partial R(\tau')}{\partial \tau'} = \left( \frac{x' - x'_0}{\tau'} \right) \left[ - \frac{2\tau'}{(T')^2} R(\tau') \right]$$

In the limit when  $T' \rightarrow \infty$ ,  $\langle \xi_0 \frac{\partial \xi}{\partial x} \rangle \rightarrow 0$  and hence,  $\langle \xi_0 \cdot \nabla \xi \rangle \xrightarrow[T' \rightarrow \infty]{} 0$ .

Similarly, we have that  $\langle \xi \cdot \nabla_0 \xi_0 \rangle \xrightarrow[T' \rightarrow \infty]{} 0$ . We continue this evaluation for higher order derivatives.

$$\langle \xi_0 \frac{\partial^2 \xi}{\partial x^2} \rangle = \frac{\partial}{\partial x'} \left[ \frac{x' - x'_0}{\tau'} \left( \frac{2\tau'}{(T')^2} \right) R(\tau') \right] = - \frac{2R(\tau')}{(T')^2} + \frac{4(x' - x'_0)^2 R(\tau')}{(T')^4}$$

This expression  $\rightarrow 0$  as  $T' \rightarrow \infty$  so that  $\langle \xi_0 \nabla^2 \xi \rangle \xrightarrow[T' \rightarrow \infty]{} 0$  and  $\langle \xi \nabla_0^2 \xi_0 \rangle \xrightarrow[T' \rightarrow \infty]{} 0$ .

In similar fashion we can show that the general cases are

$$\langle \xi_0 \nabla^n \xi \rangle \xrightarrow[T' \rightarrow \infty]{} 0, \langle \xi \nabla_0^n \xi_0 \rangle \xrightarrow[T' \rightarrow \infty]{} 0 \text{ and } \langle \nabla^m \xi \nabla_0^n \xi_0 \rangle \xrightarrow[T' \rightarrow \infty]{} 0.$$

In Brown's formalism, the expression for  $\tilde{p}_2$  when

$$p_2(\xi, \xi_0, \nabla \xi, \nabla_0 \xi_0, \dots, \nabla^n \xi, \nabla_0^n \xi_0)$$

is Gaussian is

$$\tilde{p}_2(k_1 - \beta_0, \beta_0 - \beta_1, \tilde{k}_2, -\tilde{\beta}_2, \dots) = \exp [ - (\tilde{V}_2^T \tilde{C}_2 \tilde{V}_2) / 2 ]$$

where

$$\bar{V}_2 = \begin{bmatrix} k_1 - \beta_0 \\ \beta_0 - \beta_1 \\ \bar{k}_2 \\ -\bar{\beta}_2 \\ \vdots \\ \vdots \end{bmatrix}$$

and  $\bar{C}_2$  is the covariance matrix

$$\bar{C}_2 = \begin{bmatrix} \langle \xi^2 \rangle & \langle \xi \xi_0 \rangle & \langle \xi \cdot \nabla \xi \rangle & \dots \\ \langle \xi_0 \xi \rangle & \langle \xi_0^2 \rangle & \langle \xi_0 \cdot \nabla \xi \rangle & \dots \\ \langle \nabla \xi \cdot \xi \rangle & \langle \nabla \xi \cdot \xi_0 \rangle & \langle (\nabla \xi)^2 \rangle & \dots \\ \langle \nabla_0 \xi_0 \cdot \xi \rangle & \langle \nabla_0 \xi_0 \cdot \xi_0 \rangle & \langle \nabla_0 \xi_0 \cdot \nabla \xi \rangle & \dots \\ \vdots & \vdots & \vdots & \dots \\ \vdots & \vdots & \vdots & \dots \\ \vdots & \vdots & \vdots & \dots \\ \vdots & \vdots & \vdots & \dots \end{bmatrix}$$

This form of the covariance matrix obtained by Brown is a general one. For our case with a Gaussian form for the correlation, we obtain the simpler form for the matrix

$$\bar{C}_2 = \begin{bmatrix} \sigma^2 & R & 0 & 0 & 0 & \dots \\ R & \sigma^2 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \dots \end{bmatrix}$$

by substituting our previously derived terms for the derivatives as  $T' \rightarrow \infty$ .

such that the Rayleigh parameter,  $\Sigma \geq 3$ . For smaller values, the Taylor series representation for  $\sigma^\circ$  is accurate and should be used for calculating the bistatic clutter power.

If the Rayleigh parameter dependence of the cross section is taken into account and surface shadow regions are included, then these results indicate that physical optics principles can be applied to analyze electromagnetic scattering from surfaces with an extensive range of roughness conditions.

and surface parameters are such that the Rayleigh parameter  $\Sigma \geq 3.0$ . For smaller Rayleigh parameters, the Taylor series representation for  $\sigma^\circ$  is accurate and should be used for calculations of clutter power.

## 5. CONCLUSIONS

In this report four major topics have been studied. The findings for each are summarized here. The first topic is the investigation of the conditions under which a physical optics model of the scattering from a rough surface is a valid representation. The investigation proceeded from an analysis of the magnetic field integral equation for the current density on the rough surface. For a surface with a Gaussian surface height distribution and a Gaussian surface correlation function we have demonstrated that only one sufficient condition is needed in order that physical optics be valid for calculating EM wave scattering from the rough surface. This condition is that the correlation length  $T$  be much greater than a wavelength.

The second consideration of the report is to relate that constraint to the alternative condition that the radius of curvature of the surface irregularities be large compared to the wavelength. The analysis involved the same restrictions of Gaussian surface heights and surface correlation function. For our condition ( $T/\lambda \rightarrow \infty$ ) it was shown that the consequence of these assumptions is that the surface slopes and slope derivatives are statistically independent. The conclusion that follows from these arguments is that  $T \gg \lambda$  is always a sufficient condition for  $R_c \gg \lambda$  and for most cases it is also necessary.

In the third topic we have addressed the question of the form of the scattering cross section that is present in the physical optics representation of the scattering. In the report we have shown that, despite the constraint on surface correlation, the surface can be described by a wide range of values of the Rayleigh roughness parameter. Typically in radar engineering, the asymptotic form for the cross section is used in scattering calculations. Here we have shown that different representations for the normalized cross section  $\sigma^\circ$  must be used, depending on the magnitude of the Rayleigh roughness parameter. Explicit representations for  $\sigma^\circ$  have been given for small, intermediate, and large Rayleigh roughness parameter when the surface is described by either a bivariate Gaussian distribution or a bivariate exponential distribution in height.

Finally, we examined the effect of using the different representations for  $\sigma^\circ$  in determining the diffuse power scattered in the forward direction by a bistatic radar system. The conclusion is that the conventional asymptotic form for  $\sigma^\circ$  gives accurate results ( $\pm 1$  dB) only when the system and surface parameters are



Table 3. Diffuse Scattered Power for Different  $\sigma^\circ$  Representations

Antenna Separation	$\sigma^2$	Strip Distance	$\Sigma^2$	Diffuse Scattering Power
92 km	1 m <sup>2</sup>	10 m	531.	PDIFF = 7.52.10 <sup>-13</sup> W PDIFFI = 7.57.10 <sup>-13</sup> W PDIFFX = 2.1.10 <sup>-113</sup> W
		1 km	6.67	PDIFF = 3.88.10 <sup>-12</sup> W PDIFFI = 4.16.10 <sup>-12</sup> W PDIFFX = 4.16.10 <sup>-12</sup> W
		4 km	0.795	PDIFF = 1.09.10 <sup>-12</sup> W PDIFFI = 6.75.10 <sup>-13</sup> W PDIFFX = 6.75.10 <sup>-13</sup> W
		46 km	0.4	PDIFF = 7.28.10 <sup>-15</sup> W PDIFFI = 4.41.10 <sup>-15</sup> W PDIFFX = 4.41.10 <sup>-15</sup> W
	10 m <sup>2</sup>	46 km	4.3	PDIFF = 3.65.10 <sup>-14</sup> W PDIFFI = 3.52.10 <sup>-14</sup> W PDIFFX = 3.52.10 <sup>-14</sup> W
	0.5 m <sup>2</sup>	46 km	0.2	PDIFF = 5.53.10 <sup>-15</sup> W PDIFFI = 2.27.10 <sup>-15</sup> W PDIFFX = 2.27.10 <sup>-15</sup> W
	0.1 m <sup>2</sup>	46 km	0.04	PDIFF = 2.67.10 <sup>-16</sup> W PDIFFI = 4.67.10 <sup>-16</sup> W PDIFFX = 4.67.10 <sup>-16</sup> W
	10 m <sup>2</sup>	10 m	5308.	PDIFF = 6.75.10 <sup>-12</sup> W PDIFFI = 6.75.10 <sup>-12</sup> W PDIFFX = 2.08.10 <sup>-112</sup> W
9 km	10 m <sup>2</sup>	10 m	6663.	PDIFF = 3.04.10 <sup>-9</sup> W PDIFFI = 3.04.10 <sup>-9</sup> W PDIFFX = 2.08.10 <sup>-113</sup> W
		1 km	328.	PDIFF = 3.62.10 <sup>-13</sup> W PDIFFI = 3.63.10 <sup>-13</sup> W PDIFFX = 2.08.10 <sup>-113</sup> W

contribution for two strips have been evaluated, one close to the transmitter and the second at a distance from it. These comparisons show the effect of  $\Sigma^2$  on the three types of power calculations as a number of parameters are allowed to vary widely.

In calculations of radar clutter contributions from Gaussian surfaces, the asymptotic form of  $\sigma^\circ$  is conventionally used. Examination of the results of this table indicates that the asymptotic form (physical optics plus steepest descent evaluation of integral) for  $\sigma^\circ$  will give accurate results ( $\pm 1$  dB) only if the system

#### 4. EFFECT OF $\sigma^0$ REPRESENTATION ON DIFFUSE SCATTERED POWER

In this section, actual quantitative comparisons will be made of the diffuse power reaching a receiver (see Figure 1), where the different representations for  $\sigma^0$  have been used, as discussed in Section 3. Because the computation time was prohibitive when the total diffuse power was calculated by integrating over the entire glistening surface (especially for the integral representations for  $\sigma^0$ ), we consider only the diffuse power reaching the receiver from selected strips of ground 20 m wide and parallel to the y-axis.

The effect of integrating across the surface can be seen in the results shown in Table 3. For  $\sigma^2 = 1 \text{ m}^2$  and an antenna separation of 92 km, we show the individual contributions from strips that represent a wide variation in position along the distance axis. We are concerned with the effect of the Rayleigh parameter  $\Sigma$  on the various representations. There are two main factors constituting that parameter,  $\sigma^2$  and scattering angle. In the table,  $\Sigma^2$  results are obtained, where both of these factors are allowed to vary separately. The results presented here are for a receiver height of 2500 m, transmitter height of 100 m, transmitted power of 350 W and vertical polarization. The surface heights are taken to have a Gaussian distribution and a surface correlation length  $T = 5 \text{ m}$ . For all cases, we show the three different diffuse power levels calculated using the three representations of  $\sigma^0$ : PDIFF corresponds to the asymptotic form; PDIFFI is the numerical evaluation of the integral; and PDIFFX is the result for the power series representation.

There are three basic comparisons in the table. First, for a separation of 92 km and  $\sigma^2 = 1 \text{ m}^2$  the effect of  $\Sigma^2$  is shown as the distance of the scattering strip (scattering angle) is varied. Next, for the same separation (92 km) and a fixed strip position (46 km from the transmitter)  $\Sigma^2$  is changed by varying  $\sigma^2$  ( $0.1 \text{ m}^2 \leq \sigma^2 \leq 10 \text{ m}^2$ ). Finally, the table shows a comparison of results for  $\sigma^2 = 10 \text{ m}^2$  at two antenna separations (9 and 92 km). At both separations the

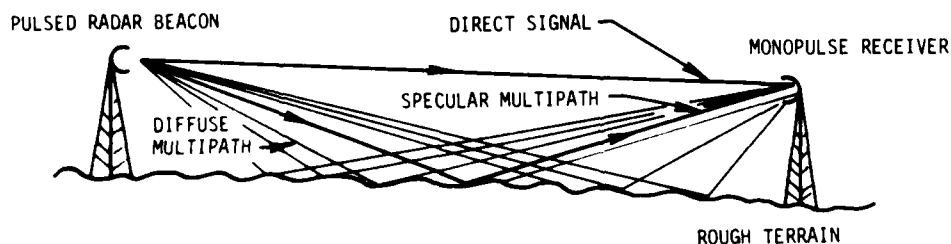


Figure 1. Rough Surface Scattering for a Bistatic Geometry

Table 2.  $J^\circ$  Values Calculated Using Different Representations

$J^\circ$ for Bivariate Gaussian			
$\Sigma^2$	POWER EXP.	ASYMPT.	INTEGRAL
96	1.0E - 13	7753	7807
20	7323	7753	8018
10	8300	7753	8298
1	3850	7753	9931
$J^\circ$ for Bivariate Exponential			
$\Sigma^2$	POWER EXP.	ASYMPT.	INTEGRAL
96		9881	9850
10	155779	9881	9985
1	7753	9881	3735
0.1	91	9881	85

The results of this table may be summarized as follows:

(i) Bivariate Gaussian Surface:

- (a) For large Rayleigh parameters ( $\Sigma^2 \geq 20$ ), both the asymptotic and integral representations are accurate and agree to within a few percent.
- (b) For intermediate and large Rayleigh parameters ( $\Sigma^2 \geq 10$ ), the integral representation is accurate (for small  $\Sigma$  values the numerical integration may have oscillation problems).
- (c) For small Rayleigh parameters ( $\Sigma^2 \leq 10$ ), only the power series expansion representation is accurate.

(ii) Bivariate Exponential Surface:

- (a) For large Rayleigh parameters ( $\Sigma^2 \geq 20$ ), both the asymptotic and integral representations are accurate and agree to within a few percent.
- (b) For all values of the Rayleigh parameter, the integral representation is accurate.
- (c) For small values of the Rayleigh parameter ( $\Sigma^2 < 1$ ), the power series expansion representation is accurate and agrees with the integral representation to within a few percent.

Brown<sup>17</sup> has identified  $J_D$  as the incoherent power scattered diffusely, and  $J_S$  is the incoherent power scattered in the specular direction ( $v_x = v_y = 0$ ). The four-fold integral given in Eq. (9a) can be reduced to a single integral if  $T \ll X, Y$ , just as was true for the Gaussian case. The result is

$$J_D = \left( \frac{4\pi}{\lambda} \right) (2\pi) \int_0^\infty J_0(v_{xy}\tau) \left[ \left( 1 + \frac{2}{3} \Sigma^2 (1 - c(\tau)) \right)^{-3/2} - \left( 1 + \frac{2}{3} \Sigma^2 \right)^{-3/2} \right] \tau d\tau \quad (10)$$

For large and small values of  $\Sigma^2$ ,  $J_S \approx 0$  so  $J^* \approx J_D$ . For large Rayleigh parameters ( $\Sigma^2 > 20$ ), a steepest descent evaluation of the integral in Eq. (10) yields the asymptotic expression for  $J_D$ :

$$J_D \approx \left( \frac{12\pi^2 T^2}{\lambda^2 \Sigma^2} \right) \exp \left[ \frac{-\sqrt{6} T}{2\sigma} \left( \frac{v_x^2 + v_y^2}{v_z^2} \right)^{1/2} \right] \quad (11)$$

where  $v_z = -(2\pi/\lambda)(\cos \theta_i + \cos \theta_s)$ . For intermediate values of  $\Sigma^2$ , the integral has to be evaluated, and in general,  $J_S \neq 0$  so  $J^* = J_D + J_S$ . For small Rayleigh parameters ( $\Sigma^2 < 1$ ), the following expression [from a Taylor expansion of the integral in Eq. (10)] can be used:

$$J_D \approx \left( \frac{4\pi^2}{\lambda^2} \right) \Sigma^2 T^2 \exp \left[ \frac{-v_{xy}^2 T^2}{4} \right] \quad (12)$$

### 3.3 Comparisons

To demonstrate quantitatively the accuracy of these various representations for  $J^*$  as a function of the Rayleigh parameter squared ( $\Sigma^2$ ), Table 2 was constructed. This table shows  $J^*$  values as a function of  $\Sigma^2$  using the different representations. Here, POWER EXP. refers to the Taylor series expansion representations [Eqs. (8) and (12)], ASYMPT. refers to the asymptotic representations [Eqs. (7) and (11)], and INTEGRAL refers to the integral representations [Eqs. (6) and (10)]. In these tables, the variance in surface height  $\sigma^2 = 10 \text{ m}^2$ , the correlation length  $T = 15 \text{ m}$ , and the azimuthal scattering angle  $\phi_s = 0.5^\circ$ . It should be noted that, since we are concerned with differences between solutions, the values shown for the intermediate exponential solutions are actually values for  $J_D$  since  $J_S$  would be nearly zero for all methods.

17. Brown, G.S. (1982) Scattering From a Class of Randomly Rough Surfaces, Radio Science 17:1274-1280.

Next, we consider the evaluation of this expression for different  $\Sigma^2$  regimes. If  $\Sigma \gg 1$ , a steepest descent evaluation of the integral in Eq. (6) reduces it to the asymptotic expression familiar to radar engineers (Barton and Ward<sup>16</sup>):

$$J^0 = \left( \frac{4\pi^2 T^2}{\lambda^2 \Sigma^2} \right) \exp \left[ \frac{-v_{xy}^2 T^2}{4 \Sigma^2} \right] \quad (7)$$

In the intermediate case, the integral has to be evaluated explicitly. When the Rayleigh parameter is small ( $\Sigma^2 < 10$ ), the following series representation can be used

$$J^0 = \left( \frac{4\pi^2 T^2}{\lambda^2} \right) e^{-\Sigma^2} \sum_{m=1}^{\infty} \frac{\Sigma^{2m}}{(m \cdot m!)} \exp \left[ \frac{-v_{xy}^2 T^2}{4m} \right] \quad (8)$$

### 3.2 Bivariate Exponential Solution

The preceding results are all for Gaussian distributed surface heights. For the case where the surface heights are described by a bivariate exponential the four-fold integral appearing in Eq. (5) may be written as follows:

$$J^0 = J_0 + J_s \quad (9)$$

where

$$J_D = \left( \frac{4\pi}{\lambda^2} \right) \left( \frac{1}{4XY} \right) \int_{-x}^x dX_1 \int_{-x}^x dX_2 \int_{-y}^y dY_1 \int_{-y}^y dY_2 \exp [iv_x(x_1 - x_2) + iv_y(y_1 - y_2)] \\ \times [\chi_2 - \lambda_2(\tau \rightarrow \infty)] \quad (9a)$$

and

$$J_s = \left( \frac{4\pi}{\lambda^2} \right) (4XY) \operatorname{sinc}^2(v_x X) \operatorname{sinc}^2(v_y Y) \left[ \left( 1 + \frac{2}{3} \Sigma^2 \right)^{-3/2} - \left( 1 + \frac{1}{3} \Sigma^2 \right)^{-3} \right] \quad (9b)$$

Since this is for an exponential distribution, the bivariate characteristic function for uncorrelated variates used in Eq. (9a) and Eq. (9b) has the form

$$\chi_2(\tau \rightarrow \infty) = \left[ 1 + (2/3) \Sigma^2 \right]^{-3/2}$$

16. Barton, D. K., and Ward, H. R. (1969) Handbook of Radar Measurement, Prentice-Hall, Inc. Englewood Cliffs, New Jersey.

$$\sigma^0 = |\beta_{pq}|^2 J^0$$

$$= \left[ \frac{\pi |\beta_{pq}|^2}{XY \lambda^2} \right] \int_{-x}^x dX_1 \int_{-x}^x dX_2 \int_{-y}^y dY_1 \int_{-y}^y dY_2 \exp[iv_x(X_1 - X_2) + iv_y(Y_1 - Y_2)] \\ \times [\chi_2(\Sigma, T) - \chi_1^*(\Sigma) \chi_1(\Sigma)]$$

Here,  $\beta_{pq}$  are the matrix elements for linear polarization states (see Ruck et al<sup>2</sup>), the rough surface has an area  $4XY$ , and

$T$  = Surface Correlation Length

$$v_x = (2\pi/\lambda) [\sin \theta_i - \sin \theta_s \cos \phi_s]$$

$$v_y = -(2\pi/\lambda) \sin \theta_s \sin \phi_s$$

$\chi_1$  = univariate characteristic function of the surface height distribution function,

$\chi_2$  = bivariate characteristic function,

$\Sigma$  = Rayleigh roughness parameter,

$$= (2\pi\sigma/\lambda) (\cos \theta_i + \cos \theta_s)$$

The forms assumed by  $\chi_1$  and  $\chi_2$  for Gaussian surface height distributions and for exponential surface height distributions have been derived previously.<sup>9</sup> For Gaussian surfaces,  $\chi_2(\tau \rightarrow \infty) = \exp(-\Sigma^2)$  and  $\chi_2(\tau \rightarrow \infty) = \chi_1 * \chi_1$ . On the other hand, for a surface described by a bivariate exponential distribution function,  $\chi_2(\tau \rightarrow \infty) = [1 + (2/3)\Sigma^2]^{-3/2}$  and  $\chi_2(\tau \rightarrow \infty) \neq \chi_1 * \chi_1$ . The only condition needed to reduce the four-fold integral to a single integration is  $T \ll X, Y$ . This criterion does not depend on the Rayleigh parameter but there are differences in the resulting forms for the two surface distributions.

### 3.1 Gaussian Solutions

For the Gaussian case, we have:

$$J^0 = \frac{8\pi^2}{\lambda^2} \int_0^\infty J_0(v_{xy}\tau) [\chi_2 - \chi_1^* \chi_1] \tau d\tau \quad (6)$$

where

$$v_{xy} = \sqrt{v_x^2 + v_y^2}$$

hence in that range  $T \gg \lambda \rightarrow R_c \gg \lambda$  and  $R_c \gg \lambda \rightarrow T \gg \lambda$ . Thus,  $T \gg \lambda$  is both necessary and sufficient for  $R_c \gg \lambda$  when we have intermediate  $\sigma/T$  values.

To summarize the above results, we can say that, under our assumptions of independent slope and slope derivatives, for a Gaussian surface and Gaussian correlation function,  $T \gg \lambda$  is always a sufficient condition for  $R_c \gg \lambda$ , and for intermediate and large slope conditions,  $T \gg \lambda$  is both necessary and sufficient for a physical optics solution to apply.

### 2.3 Shadowing

There is a final aspect to the relationship between roughness and physical optics. It is well known that as the average surface slope becomes large ( $\sigma/T \geq 1$ ), shadowing becomes more and more important. Then, the physical optics current density  $\vec{J} = 2(\vec{n} \times \vec{H}^i)$  does not truly hold everywhere on the surface. For large slopes, the correct current is obtained by multiplying the  $\sigma^o$  resulting from physical optics by a shadowing function  $S$ , which describes how much of the surface is unlit, that is, where  $J \rightarrow 0$  in the shadow regions. At present, this shadowing function has been derived rigorously only in the high frequency geometrical optics limit ( $\lambda \rightarrow 0$ ).<sup>14, 15</sup>

### 3. NORMALIZED SCATTERING CROSS SECTION

Previous reports and papers<sup>7-10</sup> discussed how a general expression for the normalized cross section  $\sigma^o$  may be derived from the Kirchhoff integral expression for the waves scattered from a rough surface. In the present report, a brief outline will be given for the determination of  $\sigma^o$  for different regimes of the Rayleigh parameter. The actual evaluation of the four-fold integral depends upon the surface height distribution function. The general expression is

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14. Sancer, M.I. (1969) Shadow-corrected electromagnetic scattering from a randomly rough surface, IEEE Trans. Antennas Propag. AP-17:577-585.
  15. Brown, G.S. (1980) Shadowing by Non-Gaussian random surfaces, Proceedings of the Second Workshop on Terrain and Sea Scatter, George Washington University, Washington, D. C.

Thus,  $T \gg \lambda$  is both necessary and sufficient for  $R_c \gg \lambda$  when  $\sigma/T \gg 1$ .

In the intermediate range of  $X$  the limiting case forms do not apply and the function  $U(1/2, 0, X)$  must be evaluated explicitly. The procedures for evaluation are outlined in Abramowitz.<sup>13</sup> To see how  $R_c$  varies in the intermediate  $X$  range we consider several cases as  $10.0 \geq X \geq 0.1$ . Over this range,  $1.5 \geq (\sigma/T) \geq 0.16$  and the evaluation of  $U(1/2, 0, X)$  leads to the result that  $0.5T \leq R_c \leq 2.5T$ . In the earlier report we evaluated  $R_c$  under the assumption that

$$\left\langle \frac{(1 + z'^2)^{3/2}}{(z''^2)^{1/2}} \right\rangle \approx \frac{\langle (1 + z'^2)^{3/2} \rangle}{\langle \sqrt{z''^2} \rangle} \approx \frac{(1 + \langle z'^2 \rangle)^{3/2}}{\sqrt{\langle z''^2 \rangle}}.$$

If we compare results of that early approach to those of the present case for intermediate slope conditions we see that for  $\sigma/T < 1$  the results are in reasonable agreement while they diverge for  $\sigma/T > 1$ . These results are summarized in Table 1.

Table 1. Results for  $R_c$  in the Intermediate Range of  $\sigma/T$  Values

X	$\sigma/T$	$\langle  K  \rangle$	$R_c$	Previous $R_c$
10	0.16	0.4/T	2.5T	1.9T
3	0.3	0.5/T	2.1T	1.2T
1.6	0.4	0.7/T	1.5T	1.1T
1	0.5	0.9/T	1.1T	1.1T
0.5	0.7	1.1/T	0.9T	1.15T
0.4	0.8	1.3/T	0.8T	1.24T
0.1	1.5	1.9/T	0.5T	2.7T

It should be noted that in a strict sense the regions of  $X$  and  $\sigma/T$  are not equivalent. For instance  $\sigma/T = 10 \rightarrow X = 0.0025$ , which is well into the small  $X$  approximation and similarly  $\sigma/T = 0.1 \rightarrow X = 25$ , which is in the large  $X$  solution region. Equivalently, for  $0.1 \leq X \leq 10$  we have  $1.5 \geq \sigma/T \geq 0.16$ . Thus, for completeness we should examine the small  $X$  solution for  $1.5 \leq \sigma/T \leq 10$  to complete the examination of how  $R_c$  behaves for intermediate  $\sigma/T$ . For that range  $R_c = 0.6T$ . Similarly, for  $0.16 \geq \sigma/T \geq 0.1$  we have  $2.3T \leq R_c \leq 3.6T$ . Thus over the entire range of intermediate  $\sigma/T$  values  $R_c$  is of the same order as  $T$  and

13. Abramowitz, M., and Stegun, I.A. (Ed.) (1964) Handbook of Mathematical Functions With Formulas, Graphs and Mathematical Tables, NBS Applied Mathematics Series No. 55.



After some manipulation we obtain

$$\begin{aligned}\langle |K| \rangle &= \left( \sqrt{\frac{2\langle z'^2 \rangle}{\pi}} \right) \left( \frac{2}{\sqrt{2\pi} \sqrt{\langle z'^2 \rangle}} \right) \left( \frac{\sqrt{\pi}}{2} \right) \left[ U\left( \frac{1}{2}, 0, 0.5 \langle z'^2 \rangle^{-1} \right) \right] \\ &= \left( \sqrt{\langle z'^2 \rangle / \langle z'^2 \rangle \pi} \right) U\left( \frac{1}{2}, 0, 0.5 \langle z'^2 \rangle^{-1} \right) \\ &= \left( \frac{1.38}{T} \right) U\left( \frac{1}{2}, 0, 0.5 \langle z'^2 \rangle^{-1} \right) \quad .\end{aligned}$$

where  $U(a, b, X)$  is the confluent hypergeometric function of the second kind with  $X \equiv 0.5 \langle z'^2 \rangle^{-1}$ .

We now want to examine the relationships for various slope regimes. Recall that  $\langle z'^2 \rangle = 2\sigma^2/T^2$  and  $\langle z''^2 \rangle = 12\sigma^2/T^4$ . Then, for small slopes ( $\sigma/T \ll 1$ ) we have  $(2 \langle z'^2 \rangle)^{-1} \rightarrow \infty$  and

$$\langle |K| \rangle \approx \left( \sqrt{\frac{\langle z''^2 \rangle}{\pi}} \right) \left( \sqrt{\langle z'^2 \rangle} \right)^{-1} \left( \frac{1}{2 \langle z'^2 \rangle} \right)^{-1/2} \approx \sqrt{\frac{2 \langle z''^2 \rangle}{\pi}} \approx \frac{2.76\sigma}{T^2}$$

for zeroth order and

$$\langle |K| \rangle \approx \frac{2.76\sigma}{T^2} \left[ 1 - 3 \left( \frac{\sigma}{T} \right)^2 \right]$$

for first order.

Thus, we have a zeroth order solution for  $R_c \equiv \left( \frac{1}{\langle |K| \rangle} \right)$

$$R_c = \frac{T^2}{2.76\sigma}$$

which is equivalent to the result obtained in Ulaby et al<sup>11</sup> for small slopes. Then, since  $\sigma/T \ll 1$  we have  $R_c > 0.36T$  and  $T \gg \lambda$  (a sufficient condition).

For large slope conditions ( $\sigma/T \gg 1$ ),  $\left( \frac{1}{2 \langle z'^2 \rangle} \right) \rightarrow 0$  and

$$\begin{aligned}\langle |K| \rangle &\approx \left( \frac{2}{\pi} \right) \sqrt{\langle z''^2 \rangle / \langle z'^2 \rangle} + O \left[ \left( \frac{1}{2 \langle z'^2 \rangle} \right) \ln \left( \frac{1}{2 \langle z''^2 \rangle} \right) \right] \\ &\approx \left( \frac{2}{\pi} \right) \sqrt{\frac{6}{T^2}} = \frac{1.6}{T}\end{aligned}$$

so

$$R_c \approx 0.6T \quad .$$

Since they are considering only small slopes, the curvature is considered to be a function of just the slope derivatives. For our applications, though, we are concerned with behavior for all surface slope magnitudes and therefore we cannot use their simplified expression for the curvature. As a result, the analysis is more complicated.

If we make use of the previously cited theorem, we can express the probability density function for the surface slopes as:

$$p(z') = \left( \frac{1}{\sqrt{2\pi} \langle z'^2 \rangle^{1/2}} \right) \exp \left\{ -\frac{z'^2}{2 \langle z'^2 \rangle} \right\}$$

and for the slope derivatives

$$p(z'') = \left( \frac{1}{\sqrt{2\pi} \langle z''^2 \rangle^{1/2}} \right) \exp \left[ -\frac{z''^2}{2 \langle z''^2 \rangle} \right]$$

In the previous section we showed that, if the surface correlation length  $T' \rightarrow \infty$ , then the slopes and slope derivatives are statistically independent with  $\langle z' z'' \rangle \rightarrow 0$ . Hence, under our assumptions we can write the joint probability density  $p(z', z'')$  as

$$p(z', z'') = \left( \frac{1}{2\pi \langle z'^2 \rangle^{1/2} \langle z''^2 \rangle^{1/2}} \right) \exp \left[ -\frac{1}{2} \left( \frac{z'^2}{\langle z'^2 \rangle} + \frac{z''^2}{\langle z''^2 \rangle} \right) \right]$$

Next, we rewrite the curvature magnitude as

$$|K| = |z'' [1 + z'^2]^{-3/2}| = |z''| [1 + z'^2]^{-3/2}$$

Then for the expected value we have

$$\langle |K| \rangle = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |z''| [1 + z'^2]^{-3/2} p(z', z'') dz' dz''$$

Integrating over  $z''$  results in

$$\langle |K| \rangle = \sqrt{\frac{2}{\pi} \langle z''^2 \rangle} \left( \frac{2}{\sqrt{2\pi} \sqrt{\langle z'^2 \rangle}} \right) \int_0^{\infty} [1 + z'^2]^{-3/2} \exp \left[ -\frac{z'^2}{2 \langle z'^2 \rangle} \right] dz'$$

$R_c$  is the average radius of curvature. This is the most generally used form for the autocorrelation since use of an exponential form leads to discontinuities at small distances.

In a previous report by Papa, Lennon, and Taylor,<sup>9</sup> a derivation was given for the mathematical representation of the average radius of curvature  $R_c$ . The rough surface is described by the equation  $z = \xi(x, y)$ . The average slope is given by

$$\langle z' \rangle = \left[ \left\langle \frac{\partial z}{\partial x} \frac{\partial z}{\partial x} \right\rangle \right]^{1/2},$$

where  $\langle \rangle$  denotes expectation value. The average radius of curvature  $R_c$  of a curve is given by

$$R_c = \frac{1}{\langle |K| \rangle} = \left\langle \frac{[1 + (z')^2]^{3/2}}{z''} \right\rangle,$$

where  $K$  is the curvature. The average of  $(z')^2$  is given by

$$\langle (z')^2 \rangle = \left\langle \frac{\partial \xi}{\partial x} \frac{\partial \xi}{\partial x} \right\rangle = - \frac{\partial^2 R(\tau=0)}{\partial \tau^2} = \frac{2\sigma^2}{T^2}$$

Also, the average of  $(z'')^2$  is given by

$$\langle (z'')^2 \rangle = \frac{\partial^4 R(\tau=0)}{\partial \tau^4} = \frac{12\sigma^2}{T^4}.$$

The results are consistent with the theorem from random processes that states that the distribution of the derivative of a normal process with zero mean and variance  $\sigma^2$  is again normal and the variance of the derivative is

$$\sigma_s^2 = -\sigma^2 \rho''(0).$$

This theorem will be used by us to examine the relationship between the radius of curvature and the correlation length in physical optics models.

Ulaby et al<sup>11</sup> examined this relation for the case where the slopes are restricted to the condition that  $(z')^2 \ll 1$ . They obtain the result that

$$R_c = \frac{T^2}{2.76\sigma}.$$

Using this simplified matrix, we obtain the result that in the limit as  $T' \rightarrow \infty$

$$\begin{aligned} & \tilde{p}_2(k_1 - \beta_0, \beta_1 - \beta_0, \vec{k}_2, -\vec{\beta}_2, \dots) \\ & \rightarrow \exp \left[ -\frac{\sigma^2}{2} \{ (k_1 - \beta_0)^2 + 2(\beta_0 - \beta_1)(k_1 - \beta_1) \frac{R}{\sigma^2} + (\beta_0 - \beta_1)^2 \} \right] \end{aligned}$$

These results now allow us to examine the individual terms of  $\Gamma_2$  in Eq. (3). Since  $\tilde{p}_2$  is independent of  $\vec{k}_2$  in the limit as  $T' \rightarrow \infty$  the terms

$$\left( \frac{\partial \tilde{g}(\Delta \vec{r}_t, \beta_0)}{\partial \Delta x} \frac{\partial \tilde{p}_2}{\partial k_{2x}} \right) \text{ and } \left( \frac{\partial \tilde{g}(\Delta \vec{r}_t, \beta_0)}{\partial \Delta y} \frac{\partial \tilde{p}_2}{\partial k_{2y}} \right) \text{ vanish. To examine the final}$$

term, we reformulate the integration as

$$\begin{aligned} I = & \iiint \int_{-\infty}^{\infty} i \beta_0 g(\xi) \exp[-i \beta_0 \xi] \tilde{p}_2 \times \\ & \exp \left[ i k'_0 \left( \frac{k_{ix} \Delta x}{k_0 T} + \frac{k_{iy} \Delta y}{k_0 T} \right) \right] d\Delta x d\Delta y d\beta_0 d\xi, \end{aligned}$$

and make an asymptotic evaluation of the  $\Delta x$  and  $\Delta y$  integrations, using the stationary phase method with  $k'_0 = k_0 T \rightarrow \infty$ . This leaves a double integration in  $\xi$  and  $\beta_0$ . The  $\beta_0$  terms in the integrand that results from the asymptotic evaluation are a standard form so that the entire process then reduces to a single integral in the complex  $\xi$ -plane. Careful analysis of the appropriate contour integration shows that the integral reduces to zero. The final result of this analysis is that for  $T/\lambda \rightarrow \infty$ ,  $\Gamma_2 \rightarrow 0$ . This implies that Eq. (2) reduces to

$$\frac{1}{2\pi} \int j_q(\beta_1) \tilde{p}_1(k_1 - \beta_1) d\beta_1 = 2 c_z^q \tilde{p}_1(k_1 - k_{iz}) \quad (4)$$

with  $q = x$  or  $y$ . The solution of Eq. (4) is given by the result

$j_q(z) = 2H_0 c_z^q \exp(ik_{iz} \xi)$ , which is the physical optics current. Thus,  $T/\lambda \rightarrow \infty$  implies that the current density on a rough surface is given by the physical optics result [ $\vec{J} = 2\vec{n} \times \vec{H}^i$ ] (for a perfectly conducting surface).

## 2.2 Relation Between $R_c$ and $T/\lambda$

We have derived the physical optics current from the magnetic field integral equation using  $T/\lambda \rightarrow \infty$ . We next show the relation between this condition and the basic condition for physical optics, namely that  $R_c \gg \lambda$ . We will show that for a rough surface with an autocorrelation function  $R = \sigma^2 e^{-\tau^2/T^2}$ , where  $\tau = [(x_1 - x_2)^2 + (y_1 - y_2)^2]^{1/2}$ , the condition  $T \gg \lambda$  implies  $R_c \gg \lambda$ , where

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## Appendix A

### An Analysis of the Magnetic Field Integral Equation

Brown<sup>A1</sup> has outlined how a general expression of the magnetic field integral equation for the current  $\vec{J}_s$  induced on a perfectly conducting rough surface by an incident magnetic field  $\vec{H}^i$  can be expanded into a form that will allow us to analyze conditions sufficient for a physical optics model to be applied to the scattering from a rough surface. The details of the analysis are presented here.

The current is given by:

$$\vec{J}_s(\vec{r}) = 2\vec{n} \times \vec{H}^i(\vec{r}) + \frac{1}{2\pi} \int_{S_0} \vec{n}(\vec{r}) \times [\vec{J}_s(\vec{r}_0) \times \nabla_0 g(|\vec{r} - \vec{r}_0|)] dS_0, \quad (A1)$$

where  $\vec{n}$  is the unit normal to rough surface  $S_0$  described by  $z_0 = \xi_0(x_0, y_0)$  and  $g(|\vec{r} - \vec{r}_0|)$  is proportional to the free space Green's function:

$$\vec{n}(\vec{r}) = [-\xi_x \vec{x} - \xi_y \vec{y} + \vec{z}] [1 + \xi_x^2 + \xi_y^2]^{-1/2}$$

$$g(|\vec{r} - \vec{r}_0|) = [\exp(ik|\vec{r} - \vec{r}_0|)] [|\vec{r} - \vec{r}_0|]^{-1},$$

A1. Brown, G. S. (1982) A stochastic Fourier transform approach to scattering from perfectly conducting randomly rough surfaces, IEEE Trans. Antennas Propag. AP-30 (No. 6):1135-1144.

where  $\vec{r}_0$  is the vector from the origin to a point on the rough surface:

$$\vec{r}_0 = x_0 \vec{u}_x + y_0 \vec{u}_y + z_0 \vec{u}_z \quad \text{and} \quad \vec{r}_{t_0} = x_0 \vec{u}_x + y_0 \vec{u}_y$$

Also,  $\xi(x, y) = z(x, y)$  is the surface height (a random variable) and  $\xi_x = \frac{\partial \xi}{\partial x}$  and  $\xi_y = \frac{\partial \xi}{\partial y}$  are the x and y components of the surface slope.

To examine the terms in the magnetic field integral equation [Eq. (A1)], the double cross product is expanded, the surface integration is converted to an integration over the  $z_0 = 0$  plane through  $dS_0 = (\sqrt{1 + \xi_{x_0}^2 + \xi_{y_0}^2}) d\vec{r}_{t_0}$  and both sides of the equation are multiplied by  $\sqrt{1 + \xi_x^2 + \xi_y^2}$  to give

$$\begin{aligned} \vec{J}(\vec{r}) = & 2\vec{n} \times \vec{H}^i(\vec{r}) + (2\pi)^{-1} \int \{ [\vec{N}(\vec{r}) \cdot \vec{\nabla}_0 g] \vec{J}(\vec{r}_0) \\ & - [\vec{N}(\vec{r}) \cdot \vec{J}(\vec{r}_0)] \vec{\nabla}_0 g \} d\vec{r}_{t_0} \end{aligned} \quad (A2)$$

where

$$\begin{aligned} \vec{J}(\vec{r}) &= (\sqrt{1 + \xi_x^2 + \xi_y^2}) \vec{J}_s(\vec{r}) \\ \vec{N}(\vec{r}) &= (\sqrt{1 + \xi_x^2 + \xi_y^2}) \vec{n}(\vec{r}) \end{aligned}$$

and

$$d\vec{r}_{t_0} = dx_0 dy_0$$

The current must always be tangential to the surface, so that

$$\vec{N}(\vec{r}_0) \cdot \vec{J}(\vec{r}_0) = 0$$

and therefore,

$$J_z(\vec{r}_0) = \xi_{x_0} J_x(\vec{r}_0) + \xi_{y_0} J_y(\vec{r}_0) \quad (A3)$$

If Eq. (A3) is substituted into Eq. (A2), the result will give two coupled integral equations for  $J_x(r)$  and  $J_y(r)$ . The equations will become uncoupled if the term  $[\vec{N} \cdot \vec{J}(\vec{r}_0)] \vec{\nabla}_0 g$  in Eq. (A2) is ignored. This is equivalent to neglecting cross polarization contributions to the current. These effects could be included in the analysis but it becomes extremely cumbersome.

In general, the electric field in the far field may be expressed in terms of the current density as follows:



$$\vec{E}_s = -i \left( \frac{k\eta_0}{4\pi} \right) g(R_0) \vec{k}_s \times \vec{k}_s \times \int \vec{J}(\vec{r}) \exp(i\vec{k}_s \cdot \vec{r}) d\vec{r}_t, \quad (A4)$$

where

$$\begin{aligned} \vec{r} &= x\vec{u}_x + y\vec{u}_y + \xi\vec{u}_z \\ \vec{k}_s &= k_{sx}\vec{u}_x + k_{sy}\vec{u}_y + k_{sz}\vec{u}_z \\ d\vec{r}_t &= dx dy \\ \eta_0 &= \text{impedance} \end{aligned}$$

and  $R_0$  is the distance from the  $z = 0$  plane to the observation point.

The statistical moments of the scattered field  $\langle E_s \rangle$  and  $\langle |E_s|^2 \rangle$  can be determined from Eqs. (A2), (A3), and (A4), where  $\langle \rangle$  denotes an ensemble average. It is possible to obtain an equation for the current  $\vec{J}_s$  in the Stochastic Fourier Transform domain by multiplying Eq. (A2) on both sides of the equation by the Fourier kernel

$$\exp[-ik_1\xi - i \sum_{n=1}^{\infty} \vec{k}_{n+1} \cdot \nabla^n \xi]$$

and then averaging over all stochastic variables using the joint probability density function relating all the heights, slopes, and slope derivatives. Performing these operations on the left-hand side of Eq. (A2) yields the result:

$$\begin{aligned} &\langle J_q(\vec{r}) \exp[-ik_1\xi - i \sum_{n=1}^{\infty} \vec{k}_{n+1} \cdot \nabla^n \xi] \rangle \\ &= \lim_{n \rightarrow \infty} (2\pi)^{-S_n} \int \dots \int \tilde{J}_q(\vec{r}_t, \beta_1, \beta_2, \dots, \beta_n) \\ &\quad \times \tilde{P}_1(k_1 - \beta_1, \vec{k}_2 - \beta_2, \dots, \vec{k}_n - \beta_n) d\beta_1 d\beta_2 \dots d\beta_n, \end{aligned}$$

where

$$\begin{aligned} S_n &= \sum_{i=1}^n i, \\ \vec{r}_t &= x\vec{u}_x + y\vec{u}_y, \end{aligned}$$

$\tilde{J}_q$  is the Stochastic Fourier transform of  $J_q(\vec{r})$ ,  $q = x$  or  $y$  and  $\tilde{P}_1(\dots)$  is the Stochastic Fourier transform of the joint probability density function ( $p d f$ ). At this point we have introduced the uncoupled current terms.

If the incident magnetic field of the surface is assumed to have the form

$$\vec{H}^i(\vec{r}) = H_0 \vec{n}_H \exp(+i\vec{k}_{it} \cdot \vec{r}_t + ik_{iz}\xi)$$

then the term  $2\vec{q} \cdot [\vec{N} \times \vec{H}^i]$  on the right-hand side of Eq. (A2) may be written

$$2\vec{q} \cdot [\vec{N}(\vec{r}) \times \vec{H}^i(\vec{r})] = 2H_0 [c_z^q + c_x^q \xi_x + c_y^q \xi_y] \exp(+i\vec{k}_{it} \cdot \vec{r}_t + ik_{iz}\xi)$$

The polarization factors are determined by the direction  $\vec{n}_H$  of the incident magnetic field:

$$\begin{aligned} c_z^q &= \vec{q} \cdot (\vec{u}_z \times \vec{n}_H) \\ c_x^q &= -\vec{q} \cdot (\vec{u}_x \times \vec{n}_H) \\ c_y^q &= -\vec{q} \cdot (\vec{u}_y \times \vec{n}_H) \end{aligned}$$

The Stochastic Fourier transform of this term may be written:

$$\begin{aligned} &\langle 2\vec{q} \cdot (\vec{N} \times \vec{H}_i) \exp[-ik_1\xi - i \sum_{n=1}^{\infty} \vec{k}_{n+1} \cdot \nabla \vec{n}_\xi] \rangle \\ &= 2H_0 \exp(+i\vec{k}_{it} \cdot \vec{r}_t) \left[ c_z^q - ic_x^q \frac{\partial}{\partial k_{2x}} - ic_y^q \frac{\partial}{\partial k_{2y}} \right] \tilde{P}_1(k_1 - k_{iz}, \vec{k}_2, \vec{k}_3, \dots) \end{aligned} \quad (A6)$$

where

$$\vec{k}_2 = k_{2x} \vec{k}_{2x} + k_{2y} \vec{k}_{2y}$$

As before, neglecting the coupled term  $[\vec{N} \cdot \vec{J}(\vec{r})] \nabla_0 \vec{g}$  on the right-hand side of Eq. (A2), the Stochastic Fourier Transform of the integral in Eq. (A2) may be written:

$$\begin{aligned}
& \left\langle \int \left[ -\xi_x \frac{\partial g}{\partial x_0} - \xi_y \frac{\partial g}{\partial y_0} + \frac{\partial g}{\partial \xi_0} \right] J_q(\vec{r}_{t_0}, \xi_0, \nabla_0 \xi_0, \dots) \exp \left[ -ik_1 \xi - i \sum_{n=1}^{\infty} \vec{k}_{n+1} \cdot \vec{\nabla}^n \xi \right] d\vec{r}_{t_0} \right\rangle \\
& = \int \dots \int \left[ -\xi_x \frac{\partial g}{\partial x_0} - \xi_y \frac{\partial g}{\partial y_0} + \frac{\partial g}{\partial \xi_0} \right] J_q(\vec{r}_{t_0}, \xi_0, \nabla_0 \xi_0, \dots) \exp \left[ -ik_1 \xi - i \sum_{n=1}^{\infty} \vec{k}_{n+1} \cdot \vec{\nabla}^n \xi \right] \\
& \quad \times P_2(\xi, \xi_0, \nabla \xi, \nabla_0 \xi_0, \dots) d\vec{r}_{t_0} d\xi d\xi_0 \dots, \quad (A7)
\end{aligned}$$

where  $P_2$  is the two point pdf of the variates. The  $\xi$ -integration can be written as a convolution of the  $\xi$ -Fourier transforms of the Green's function derivatives and  $P_2(\dots)$ .

By using the relations

$$\tilde{g}(\Delta \vec{r}_t, \beta_0) = \int g(\Delta \vec{r}_t, \xi) \exp(+i\beta_0 \xi) d\xi$$

and

$$\tilde{g}_\xi(\Delta \vec{r}_t, \beta_0) = \int \left( \frac{\partial g(\Delta \vec{r}_t, \xi)}{\partial \xi} \right) \exp(i\beta_0 \xi) d\xi$$

and making the substitution  $\Delta \vec{r}_t = \vec{r}_t - \vec{r}_{t_0}$ , Eq. (A7) may be written

$$\begin{aligned}
& \left\langle \int \left[ -\xi_x \frac{\partial g}{\partial x_0} - \xi_y \frac{\partial g}{\partial y_0} + \frac{\partial g}{\partial \xi_0} \right] J_q(\vec{r}_{t_0}, \xi_0, \nabla_0 \xi_0, \dots) \right. \\
& \quad \left. \exp \left[ -ik_1 \xi - i\beta_0(\xi_0 - \xi) - i \sum_{n=1}^{\infty} \vec{k}_{n+1} \cdot \vec{\nabla}^n \xi \right] d\vec{r}_{t_0} \right\rangle \\
& = \frac{1}{2\pi} \int \dots \int \left[ -\xi_x \frac{\partial \tilde{g}(\Delta \vec{r}_t, \beta_0)}{\partial x_0} - \xi_y \frac{\partial \tilde{g}(\Delta \vec{r}_t, \beta_0)}{\partial y_0} - \tilde{g}_\xi(\Delta \vec{r}_t, \beta_0) \right] J_q(\vec{r}_{t_0}, \xi_0, \nabla_0 \xi_0, \dots) \\
& \quad \times \exp \left[ -i\beta_0 \xi_0 - i \sum_{n=1}^{\infty} \vec{k}_{n+1} \cdot \vec{\nabla}^n \xi \right] \tilde{P}_2(k_1 - \beta_0, \xi_0, \nabla \xi, \nabla_0 \xi_0, \dots) d\vec{r}_{t_0} d\beta_0 d\xi_0 \dots \quad (A8)
\end{aligned}$$

where  $\tilde{P}_2$  is the Fourier transform of  $P_2(\dots)$  with respect to  $\xi$ .

The integration over  $\xi_0$  may be expressed as a convolution with a shifted argument due to the exponential factor in Eq. (A8). The  $\nabla_0 \xi_0, \nabla_0^2 \xi_0, \dots$  integrations may be written as convolution with no shift in the argument, and the integrations over  $\nabla \xi, \nabla_\xi^2, \dots$  may be written as simple Fourier transforms.

If these transforms are performed, Eq. (A8) becomes

$$\begin{aligned}
& \langle \left[ -\xi_x \frac{\partial \tilde{g}}{\partial x_0} - \xi_y \frac{\partial \tilde{g}}{\partial y_0} + \frac{\partial \tilde{g}}{\partial \xi_0} \right] \dots \rangle \\
& = \lim_{n \rightarrow \infty} (2\pi)^{-2-S_n} \int \dots \int \left[ -i \frac{\partial \tilde{g}}{\partial x_0} \frac{\partial}{\partial k_{2x}} - i \frac{\partial \tilde{g}}{\partial y_0} \frac{\partial}{\partial k_{2y}} - \tilde{g}_\xi(\Delta \vec{r}_t, \beta_0) \right] \\
& \quad \times \tilde{J}_q(\vec{r}_t, \beta_1, \beta_2, \dots, \beta_n) \tilde{P}_2(k_1 - \beta_0, \beta_0 - \beta_1, \vec{k}_2, -\beta_2, \dots; \Delta \vec{r}_t) d\vec{r}_t d\beta_0 d\beta_1 \dots d\beta_n
\end{aligned} \tag{A9}$$

where

$$\begin{aligned}
\tilde{J}_q(\vec{r}_t, \beta_1, \beta_2, \dots) & = \int \dots \int J_q(\vec{r}_t, \xi_0, \nabla_0 \xi_0, \dots, \nabla_0^n \xi_0) \\
& \quad \times \exp \left[ -i\beta_1 \xi_0 - i \sum_{n=1}^{\infty} \tilde{\beta}_{n+1} \cdot \nabla_0^n \xi_0 \right] d\xi_0 d\nabla_0 \xi_0 \dots d\nabla_0^n \xi_0
\end{aligned}$$

and  $\tilde{P}_2$  is a double Fourier Transform with respect to both  $\xi$  and  $\xi_0$ .

$$\begin{aligned}
\tilde{P}_2(k_1 - \beta_0, \beta_0 - \beta_1, k_2, -\beta_2, \dots, \Delta \vec{r}_t) & = \iint P_2(\xi, \xi_0, \nabla \xi, \nabla_0 \xi_0, \dots; \Delta \vec{r}_t) \\
& \quad \times \exp \left[ -i(k_1 - \beta_0)\xi - i(\beta_0 - \beta_1)\xi_0 - i \sum_{n=1}^{\infty} \vec{k}_{n+1} \cdot \nabla^n \xi \right. \\
& \quad \left. + i \sum_{n=1}^{\infty} \tilde{\beta}_{n+1} \cdot \nabla_0^n \xi_0 \right] d\xi d\xi_0 d\nabla \xi d\nabla_0 \xi_0 \dots
\end{aligned}$$

The Stochastic Fourier Transform of Eq. (A2) may now be determined by equating Eq. (A5) to the sum of Eqs. (A6) and (A9). With this and the substitution  $\Delta \vec{r}_t = \vec{r}_t - \vec{r}_{t_0}$ , the following result may be obtained:

$$\begin{aligned}
& \lim_{n \rightarrow \infty} \frac{1}{(2\pi)^{S_n}} \int \dots \int \tilde{J}_q(\vec{r}_t, \beta_1, \beta_2, \dots, \beta_n) \tilde{P}_1(k_1 - \beta_1, \vec{k}_2 - \beta_2, \dots, \vec{k}_n - \beta_n) d\beta_1 d\beta_2 \dots d\beta_n \\
& = 2H_0 \exp(+i\vec{k}_t \cdot \vec{r}_t) \left[ c_z^q - ic_x^q \frac{\partial}{\partial k_{2x}} - ic_y^q \frac{\partial}{\partial k_{2y}} \right] \tilde{P}_1(k_1 - k_{1z}, \vec{k}_2, \vec{k}_3, \dots) \\
& \quad + \lim_{n \rightarrow \infty} (2\pi)^{-2-S_n} \int \dots \int \left[ -i \frac{\partial \tilde{g}(\Delta \vec{r}_t, \beta_0)}{\partial \Delta x} \frac{\partial}{\partial k_{2x}} - i \frac{\partial \tilde{g}(\Delta \vec{r}_t, \beta_0)}{\partial \Delta y} \frac{\partial}{\partial k_{2y}} + \tilde{g}_\xi(\Delta \vec{r}_t, \beta_0) \right] \\
& \quad \times \tilde{J}_q(\vec{r}_t - \Delta \vec{r}_t, \beta_1, \beta_2, \dots, \beta_n) \\
& \quad \times \tilde{P}_2(k_1 - \beta_0, \beta_0 - \beta_1, \vec{k}_2, -\beta_2, \dots, \vec{k}_n, -\beta_n, \Delta \vec{r}_t) d\Delta \vec{r}_t d\beta_0 \dots d\beta_n \quad (A10)
\end{aligned}$$

The first term of the right-hand side of Eq. (A10) has a factor  $\exp(+i\vec{k}_{it} \cdot \vec{r}_t)$ . This implies that the current may be written in the form

$$J_q(\vec{r}_t, \beta_1, \vec{\beta}_2, \dots) = j_q(\beta_1, \vec{\beta}_2, \dots) \exp(+i\vec{k}_{it} \cdot \vec{r}_t) \quad .$$

When this is substituted into Eq. (A10), the following equation for  $j_q$  may be obtained

$$\begin{aligned} \lim_{n \rightarrow \infty} \left( \frac{1}{2\pi} \right)^{S_n} \int \dots \int j_i(\beta_1, \vec{\beta}_2, \dots) & \left[ \tilde{P}_1(k_1 - \beta_1, \vec{k}_2 - \vec{\beta}_2, \dots, \vec{k}_n - \vec{\beta}_n) \right. \\ & \left. - \Gamma_2(\beta_1, k_1, \vec{\beta}_2, \vec{k}_2, \dots) \right] d\beta_1 d\vec{\beta}_2 \dots d\vec{\beta}_n \\ & = 2H_0 \left[ c_z^q - i c_x^q \frac{\partial}{\partial k_{2x}} - i c_y^q \frac{\partial}{\partial k_{2y}} \right] \tilde{P}_1(k_1 - k_{iz}, \vec{k}_2, \dots, \vec{k}_n) \quad . \end{aligned}$$

Here, the quantity  $\Gamma_2$  is given by

$$\begin{aligned} \Gamma_2(\beta_1, k_1, \vec{\beta}_2, \vec{k}_2, \dots, \vec{k}_n) \\ & = \frac{1}{(2\pi)^2} \iint \left[ -i \frac{\partial \tilde{g}(\Delta \vec{r}_t, \beta_0)}{\partial \Delta x} \frac{\partial}{\partial k_{2x}} - i \frac{\partial \tilde{g}(\Delta \vec{r}_t, \beta_0)}{\partial \Delta y} \frac{\partial}{\partial k_{2y}} + \tilde{g}_\xi(\Delta \vec{r}_t, \beta_0) \right] \\ & \times \tilde{P}_2(k_1 - \beta_0, \beta_0 - \beta_1, \vec{k}_2, \dots) \exp(i\vec{k}_{it} \cdot \Delta \vec{r}_t) d\Delta \vec{r}_t d\beta_0 \quad . \end{aligned}$$

This form of the integral equation describing the current is one that is amenable to further analysis of its component terms. The term-by-term analysis leads to determination of conditions for physical optics solutions.

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